IX. *On Infinite Series. By* Edward Waring, *M. D. F. R. S.* Lucaſian *Profeſſor of Mathematics in the Univerſity of* Cambridge.

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1. MERCATOR firſt publiſhed the contiuation of the common method of diviſion to an infinite ſeries of terms proceding according to the dimenſions of a variable quantity; NEWTON did the ſame for the common method of extraction of roots. Dr. BARROW before applied the ſame principles in ſome eaſy examples to find the aſymptotes of curves.

2. The methods of diviſion and extraction of roots were long before taught; but the continuation of them *in infinitum* would have been uſeleſs, as the areæ of curves, whoſe ordinates are $ax^{\frac{n}{m}}$ (where x denotes the abſciſs, and a, n, and m invariable quantities) had not been diſcovered many years before the time of MERCATOR's Publication, and conſequently it would have been of little uſe to transform an ordinate or fluxion, whoſe area or fluent is unknown, into another form, of which the area, &c. is equally unknown.

3. Sir ISAAC NEWTON extended the rule for raiſing a binomial (to any affirmative power) to negative powers, the extraction of roots and fractional indexes, by applying the law of the ſeries for affirmative powers to them, and continuing it *in infinitum*. M. DE MOIVRE extracted the root, &c. of a multinomial by a ſeries of a ſimilar nature; but theſe methods

will

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will only apply in the moſt ſimple caſes, when not more than one root is to be extracted. In every complicate caſe (*viz.* the extraction of roots of quantities which involve the roots of compound quantities) of irrational quantities, recourſe muſt be had to the old methods of multiplication, diviſion, and extraction of roots.

4. If a root of a complicate irrational quantity be required by a ſeries proceeding according to the dimenſions of x; firſt reduce all the irrational quantities contained under the root by multiplication, diviſion, and extraction of roots into ſerieſes proceeding according to the dimenſions of x, ſo that the terms of the leaſt dimenſions be conſtituted firſt, if an aſcending ſeries be required, and ſo on; and the contrary, if a deſcending; then add the ſeveral ſums together, and extract the root of the reſulting ſum by a ſeries which proceeds according to the dimenſions of x, and it will be the root required.

Ex. Let the value of the quantity $\sqrt[3]{}\,(x^{-2}+a+\overline{b+cx}\times\sqrt[m]{}\,(d+ex+\sqrt{}\,(f+gx^2))+\sqrt{}\,(b+kx^2))$ be required by an aſcending ſeries; firſt, extract the innermoſt root $\sqrt{}\,(f+gx^2)=f^{\frac{1}{2}}+\frac{gx^2}{2f^{\frac{1}{2}}}-$ &c. which add to the quantity $d+ex$ contained under the ſame root (m), and place the terms of the ſum according to the dimenſions of x, and it becomes $(d+f^{\frac{1}{2}})+ex+\frac{gx^2}{2f^{\frac{1}{2}}}-$ &c. of which extract the (m) root, and there reſults

$$(d+f^{\frac{1}{2}})^{\frac{1}{m}}+\frac{1}{m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times ex+\left.\begin{array}{r}\frac{1}{m}\times\frac{1-m}{2m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-2m}{m}}\times e^2\\ +\frac{1}{m}\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times\frac{g}{2f^{\frac{1}{2}}}\end{array}\right\}x^2+$$

&c. multiply this ſeries into $b+cx$, and thence is derived

$$\left.\begin{array}{l} b\times\overline{d+f^{\frac{1}{2}}}^{\frac{1}{m}}+\frac{1}{m}b\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times e \\ \qquad + \quad c\times\overline{d+f^{\frac{1}{2}}}^{\frac{1}{m}} \end{array}\right\} x+\&c.=\mathrm{P};$$ extract the root $\sqrt{b+kx^2}=b^{\frac{1}{2}}+\frac{kx^2}{2b^{\frac{1}{2}}}+\&c.=\mathrm{Q}$; add the three quantities P, Q, and $x^{-2}+a$ contained under the ſame root $\sqrt[3]{}$ together, and the ſeries reſulting, whoſe terms are conſtituted according to the dimenſions of x, will be $x^{-2}+(a+b\times\overline{d+f^{\frac{1}{2}}}^{\frac{1}{m}}+b^{\frac{1}{2}}=\mathrm{A})+(\frac{1}{m}b\times\overline{d+f^{\frac{1}{2}}}^{\frac{1-m}{m}}\times e+c\times\overline{d+f^{\frac{1}{2}}}^{\frac{1}{m}}=\mathrm{B})\,x+\&c.$; of which extract the cube root $\sqrt[3]{}$, and it will be $x^{-\frac{2}{3}}+\frac{1}{3}\mathrm{A}x^{\frac{4}{3}}+\frac{1}{3}\mathrm{B}x^{\frac{7}{3}}+\&c.$ the root required.

5. The principal uſe of reducing quantities into ſeries proceeding according to the dimenſions of the variable quantity is as before mentioned for finding the area of a curve from its ordinate; or, which correſponds, the integral from its naſcent or evaneſcent increment; but the ſerieſes deduced ſhould converge, otherwiſe from them cannot be found the area or integral. In the Meditationes Analyticæ a method was firſt publiſhed of finding when theſe ſeries will converge and when not, *e. g.* the ſeries $a+bx+cx^2+dx^3+\&c.=\int(\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+\&c.)^m\times\dot{x}=\mathrm{P}$ will converge when (x) either affirmative or negative is leſs than the leaſt root (α) of the equation $\mathrm{A}+\mathrm{B}x+\mathrm{C}x^2+\mathrm{D}x^3+\&c.=0$, if the roots are poſſible. A ſimilar rule is given when ſome of the roots are impoſſible. The ſeries will diverge when x is greater than α, and the caſes are given in which it will converge when $x=\alpha$. The ſeries deſcending according to the dimenſions of x will converge when x is greater than the greateſt root of the equation, &c. Theſe principles are fur-

ther applied in the ſame Book to complicate irrational algebraical functions of x, &c.

Hence moſt commonly the ſeries for the area contained between two ordinates, or integral between two different increments deduced by the common method will diverge; on which account, in the ſame Book, is given a method by interpolation of finding the area or integral contained between any two different values of x by converging ſeries, if the area, &c. is finite.

6. To find whether a given value $(+a)$ is leſs than the leaſt affirmative or negative root (x) of a given algebraical equation $A+Bx+Cx^2+Dx^3+\&c.=0$, if all its roots are poſſible; transform the equation into another, whoſe root z is the reciprocal of the root $x=\frac{1}{z}$ of the given equation, and for z in the reſulting equation write reſpectively $v+a$ and $v-a$; and if from the former ſubſtitution all the terms become negative or affirmative, and from the latter they become alternately negative and affirmative, then will a be leſs than the leaſt root of the given equation. If in the ſame manner, in the given equation for x be ſubſtituted $v+a$ and $v-a$, and the terms reſult as before, then will (a) be greater than the greateſt root affirmative or negative of the given equation.

7. When the integral of an algebraical quantity, whoſe increments are finite, is required; firſt, by the method given in Medit. Analyt. inveſtigate the integral in finite terms, if it can be expreſſed by them; but if not, reduce it into infinite ſerieſes of which the integral of each of the terms can be found, and alſo the ſerieſes for finding the integral contained between the two different given values of the variable quantity may converge.

Serieſes of this kind have been given in the Medit. Analyt. and innumerable of a like kind may be added for finding integrals by converging ſerieſes either aſcending or deſcending, of which the given increments are either finite or evaneſcent.

7. 2. It may be obſerved, that generally the particular caſe of which the increments are naſcent or evaneſcent may be deduced from the general, in which the increments are finite; and conſequently in many caſes the general will, *mutatis mutandis*, correſpond to the particular; *e. g.* 1. the integral cannot be expreſſed in finite algebraical terms, when any factor in the denominator of the increment has not a ſucceſſive correſpondent one; which is analogous to the caſe of the ſimple diviſor in the denominator of a fluxion publiſhed in the Quadrature of Curves. 2. Nor can it be expreſſed by the above-mentioned terms, when the dimenſions of the variable quantity in the denominator exceed its dimenſions in the numerator by unity, which correſponds to a ſimilar caſe in fluxions firſt given in Medit. Analyt. To theſe may ſeveral others be added.

8. Let the fluent of the fluxion $(A+Bx^n+Cx^{2n}+\&c.)^m \times x^{\theta-1}\dot{x} = ax^\theta + bx^{\theta+n} + cx^{\theta+2n} + \&c. = (A+Bx^n+Cx^{2n}+\&c.)^{m+1} \times \overline{\alpha x^\theta + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.} = a'x^\lambda + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c. = (A+Bx^n+Cx^{2n}+\&c.)^{m+1} \times \overline{\alpha' x^{\lambda'} + \beta' x^{\lambda'-n} + \gamma' x^{\lambda'-2n} + \&c.}$

1. If the infinite aſcending ſeries $\alpha x^\theta + \beta x^{\theta+n} + \gamma x^{\theta+2n} + \&c.$ converges, then will the ſeries $ax^\theta + bx^{\theta+n} + cx^{\theta+2n} + \&c.$ converge, and nearly in the ſame ratio; and *vice verſâ*, if the former diverges, the latter will alſo diverge. In the ſame manner if the infinite deſcending ſeries $\alpha' x^{\lambda'} + \beta' x^{\lambda'-n} + \gamma' x^{\lambda'-2n} + \&c.$ converges, then will the ſeries $a'x^\lambda + b'x^{\lambda-n} + c'x^{\lambda-2n} + \&c.$ converge; and if the former diverges, then will the latter alſo diverge; and in all the caſes nearly in the ſame ratio, except only when their convergency is the leaſt.

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2. If the ſeries $ax^{\theta}+bx^{\theta+n}+cx^{\theta+2n}+\&c.$ converges, then will the ſeries $a'x^{\lambda}+b'x^{\lambda-n}+c'x^{\lambda-2n}+\&c.$ diverge, unleſs in caſes of the ſloweſt convergency, where $x=\pm v\sqrt[r]{\pm 1}$, and all the roots of the given equation are of the formula $\pm v\sqrt[s]{\pm 1}$; when $x=\pm v\sqrt[r]{\pm 1}$, the ſucceſſive terms of the infinite ſeries deduced from algebraical quantities by the preceding method will ultimately, that is, at an infinite diſtance come more near to a ratio of equality with each other than any aſſignable difference.

3. If the fluxion be $(A+Bx^{n}+Cx^{2n}\;\ldots\; x^{rn})^{m}\times x^{\theta-1}\dot{x}$, then will $\lambda=mrn+\theta$, and $\lambda'=\theta-rn$.

Many more propoſitions concerning infinite ſeries and their convergency are given in the Medit. Analyt.

4. Let the fluxion be $\overline{a+bx^{n}}^{m}\times x^{h}\dot{x}$, of which find the fluent in a ſeries aſcending according to the dimenſions of x^{n}, and it will be

$$a^{m}\times x^{h+1}\left(\frac{1}{h+1}+\frac{m}{h+n+1}\times\frac{b}{a}x^{n}+m\,.\,\frac{m-1}{2(h+2n+1)}\times\frac{b^{2}}{a^{2}}x^{2n}+m\,.\,\frac{m-1}{2}\,.\,\frac{m-2}{3(h+3n+1)}\times\frac{b^{3}}{a^{3}}x^{3n}+\&c.\right)=\frac{b^{m}x^{mn+h+1}}{mn+h+1}+$$

$$\frac{mb^{m-1}ax^{mn+h+1-n}}{mn+h+1-n}+m\,.\,\frac{m-1}{2}\times\frac{b^{m-2}a^{2}x^{mn+h+1-2n}}{mn+h+1-2n}+\&c.=\overline{a+bx^{n}}^{m+1}$$

$$\times\left(\frac{x^{h+1}}{\overline{h+1}a}-\frac{\overline{m+1}(n)+h+1}{\overline{h+1}\,.\,\overline{h+1+n}}\times\frac{b}{a^{2}}x^{h+n+1}+\frac{(\overline{m+1}(n)+h+1).(\overline{m+2}(n)+h+1)}{\overline{h+1}\,.\,\overline{h+1+n}\,.\,\overline{h+1+2n}}\right.$$

$$\times\frac{b^{2}}{a^{3}}\times x^{h+2n+1}-\frac{(\overline{m+1}\,.\,n+h+1)\,.\,(\overline{m+2n}+h+1)\,.\,(\overline{m+3n}+h+1)}{\overline{h+1}\,.\,\overline{h+1+n}\,.\,\overline{h+1+2n}\,.\,\overline{h+1+3n}}\times$$

$$\left.\frac{b^{3}}{a^{4}}x^{h+3n+1}+\&c.\right)=(bx^{n}+a)^{m+1}\times\left(\frac{1}{(mn+h+1)b}\times x^{h+1-n}-\right.$$

$$\frac{\overline{h+1-n}a}{\overline{mn+h+1}\,.\,\overline{mn+h+1-n}b^{2}}\times x^{h+1-2n}+$$

$$\frac{\overline{h+1-n}\times\overline{h+1-2n}\times a^{2}}{\overline{mn+h+1}\times\overline{mn+h+1-n}\times\overline{mn+h+1-2n}\times b^{3}}\times x^{h+1-2n}-\&c.\ (L).$$

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firſt ſeries, which aſcends according to the dimenſions of x^n, terminates, when $\frac{mn+h+1}{n}$ is a whole negative number; the ſecond, which deſcends according to the dimenſions of the ſame quantity x^n, terminates when $\frac{h+1}{n}$ is a whole affirmative number. The fluent will terminate both ways when $\frac{h+1}{n}$ is a whole affirmative and m a negative number greater than $\frac{h+1}{n}$.

5. When m is a whole number or $=0$, the fluent $\int \overline{a+bx^n}^{m} x^h \dot{x}$ can always be found in finite terms of x, or in the above-mentioned finite terms, together with the log. of x; which appears from reducing $\overline{a+bx^n}^{m}$ into ſimple terms $a^m + ma^{m-1}bx^n + m \cdot \frac{m-1}{2} a^{m-2}b^2x^{2n} + \&c.$, and multiplying them into $x^h\dot{x}$, and finding the fluents of the reſulting fluxions: but the ſeries found by the preceding method will not always terminate when m is a whole number, and the ſeries findable as above mentioned; when properly corrected, it may be rendered findable, or, which means the ſame thing, the ſeries may be made to terminate.

6. When the ſeries which expreſſes the fluent of a fluxion terminates, we may begin either from one end of the ſeries, or the other; for example, in finding the fluent of the fluxion $\overline{a+bx^n}^{m} \times x^h\dot{x}$, either aſſume the ſeries $\overline{a+bx^n}^{m+1} \times (\alpha x^{h+1} + \beta x^{h+n+1} + \gamma x^{h+2n+1} + \&c.)$ or the ſeries $\overline{a+bx^n}^{m+1} \times (A + Bx^n + Cx^{2n} + \&c.)$; the former, as is before mentioned, terminates

when $\frac{mn+b+1}{n}$ is a whole negative number; the latter when $\frac{b+1}{n}$ is a whole affirmative number.

In like manner aſſume for a deſcending ſeries $(bx^n+a)^{m+1} \times (\alpha' x^{b+1-n} + \beta' x^{b+1-2n} + \gamma' x^{b+1-3n} + \&c.)$, or $(bx^n+a)^{m+1} \times (Ax^{-\overline{m+1}n} + Bx^{-\overline{m+2}n} + Cx^{-\overline{m+3}n} + \&c.)$; the former will terminate when $\frac{b+1}{n}$ is a whole affirmative; the latter when $\frac{mn+b+1}{n}$ is a whole negative number.

It appears, therefore, that a ſeries will terminate equally by an aſcending or deſcending ſeries; and the end of the one aſcending ſeries is the beginning of its correſpondent deſcending one. All theſe ſerieſes, which do not terminate, proceed on *in infinitum*; one term in the former ſeries becomes infinite, when $b+1+zn=0$; and in the latter (L) when $mn+b+1-zn=0$, z being a whole affirmative number.

10. It has been obſerved in the Medit. Analyt. that if ſome quantities contained in the given irrational ones are much leſs or greater than the reſt, it may be preferable in the former caſe to reduce them into ſerieſes not proceeding according to the dimenſions of x, but according to the dimenſions of thoſe quantities; and in the latter caſe according to the reciprocal dimenſions of them; and particularly ſo if the fluent or integral of the terms of the reſulting ſerieſes can be found in finite terms, or by tables already calculated.

From ſimilar principles to thoſe before given may be found when the reſulting ſeries will converge, and when not.

This method will in many problems be uſeful, when the value of a near approximate is known.

Of this caſe I ſhall ſubjoin a few examples, of which ſome have been already publiſhed in the Medit. Analyt.

Ex. 1. Let the ſine and coſine of a given arc A be reſpectively s and c; then will the ſine and coſine of an arc $A+e$, where e bears a very ſmall ratio to A, be reſpectively $s+\frac{ce}{r}-\frac{s}{1\,.\,2r^2}e^2-\frac{ce^3}{1\,.\,2\,.\,3r^3}+\frac{s}{1\,.\,2\,3\,.\,4r^4}e^4+$ &c. (the ſigns proceed by pairs alternately negative and affirmative) $=s\left(1-\frac{1}{1\,.\,2r^2}\times e^2+\frac{1}{1\,.\,2\,.\,3\,.\,4r^4}\times e^4-\&c.\right)+c\left(\frac{e}{r}-\frac{e^3}{1.2.3r^3}+\frac{e^5}{1.2.3.4.5r^5}-\&c.\right)$; and $c-\frac{s}{r}e-\frac{1}{1\,.\,2r^2}ce^2+\frac{1}{1\,.\,2\,.\,3r^3}\times se^3+$ &c. (the ſigns proceed as before by pairs alternately negative and affirmative) $=c\left(1-\frac{1}{1\,.\,2r^2}e^2+\frac{1}{1\,.\,2\,.\,3\,.\,4r^2}e^4-\&c.\right)-\frac{s}{r}\left(e-\frac{1}{2\,.\,3r^2}e^3+\frac{1}{2\,.\,3\,.\,4\,.\,5r^4}e^5-\&c.\right)$

This ſeries can alſo eaſily be derived from NEWTON's ſeries by plane trigonometry, and will converge much ſwifter than the ſeries $A-\frac{A^3}{1\,.\,2\,.\,3r^2}+\frac{A^5}{2.3.4.5r^4}-$ &c. &c. if e bears a ſmall proportion to A.

Ex. 2. Let e be a ſmall quantity in proportion to t, and the given fluxion $\frac{\dot{t}}{r^2+(t+e)^2}=\frac{\dot{t}}{r^2+t^2}-\frac{2te+e^2}{(r^2+t^2)^2}\dot{t}+\frac{(2te+e^2)^2}{(r^2+t^2)^3}\times\dot{t}-\frac{(2te+e^2)^3}{(r^2+t^2)^4}\times\dot{t}+\&c.=\frac{\dot{t}}{r^2+t^2}-\frac{2t\dot{t}}{(r^2+t^2)^2}\times e-\frac{(\overline{r^2+t^2}-2^2\times t^2)\dot{t}}{(r^2+t^2)^3}\times e^2+2t\times\frac{(2\overline{r^2+t^2}-2^2t^2)\dot{t}}{(r^2+t^2)^4}\times e^3+$ &c. (P); the co-efficient of a term $\dot{t}\times e^{2m}$ of this ſeries will be a fraction whoſe numerator is $\overline{r^2+t^2}^{m}-\overline{m+1}\,.\,\frac{m}{2}\times 2^2\times t^2\times\overline{r^2+t^2}^{m-1}+\overline{m+2}\,.\,\frac{m+1}{2}\,.\,\frac{m}{3}\,.\,\frac{m-1}{4}\times 2^4\times t^4\times\overline{r^2+t^2}^{m-2}-\overline{m+3}\,.\,\frac{m+2}{2}\,.\,\frac{m+1}{3}\,.\,\frac{m}{4}\,.\,\frac{m-1}{5}\,.\,\frac{m-2}{6}$

$\times 2^6 \times t^6 \times \overline{r^2+t^2}^{\,n-3}$ + &c., and denominator $(r^2+t^2)^{2m+1}$; and the co-efficient of a term $\dot{t} \times e^{2m+1}$ will be a fraction whoſe numerator is $2t \times (\overline{m+1} \,.\, \overline{r^2+t^2}^{\,m} - \overline{m+2} \,.\, \frac{m+1}{2} \,.\, \frac{m}{3} \times 2^2 \times t^2 \times \overline{r^2+t^2}^{\,n-1} + \overline{m+3} \,.\, \frac{m+2}{2} \,.\, \frac{m+1}{3} \,.\, \frac{m}{4} \,.\, \frac{m-1}{5} \times 2^4 t^4 \times \overline{r^2+t^2}^{\,m-2} -$ &c.) and denominator $(r^2+t^2)^{2m+2}$; m being a whole number. The continuation of theſe ſerieſes is too evident to need enunciation, as muſt generally be the caſe when the number of factors contained in the ſucceſſive terms continues the ſame or increaſes in arithmetical progreſſion; and the factors themſelves increaſe or diminiſh in an arithmetical progreſſion: the terms proceed alternately + and − by pairs.

2. 2. Let the given fluxion be $\frac{\dot{e}}{r^2+(t+c)^2} = \frac{\dot{e}}{r^2+t^2} - \frac{\overline{2te+e^2} \times \dot{e}}{(r^2+t^2)^2} + \frac{(2te+e^2)^2 \times \dot{e}}{(r^2+t^2)^3} - \&c. = \frac{\dot{e}}{r^2+t^2} - \frac{2t\dot{e}}{(r^2+t^2)^2} \times e - \frac{(r^2+t^2-2^2t^2)\dot{e}}{(r^2+t\ )^3} \times e^2 + \&c.$ (Q); this ſeries becomes the ſame as the preceding by ſubſtituting in it $\dot{e}$ for $\dot{t}$; in this ſeries e is conſidered as variable, in the preceding t.

The fluent of the former ſeries (P) is $\int \frac{\dot{t}}{r^2+t^2} + \frac{e}{r^2+t^2} - \frac{t}{(r^2+t^2)^2} \times e^2 - \frac{(r^2+t^2-2^2t^2)}{3(r^2+t^2)^3} \times e^3 -$ &c.; the fluent of the latter ſeries (Q) is the ſame as of the former (P) except the firſt term $\int \frac{\dot{t}}{r^2+t^2}$: the co-efficient of the term e^h in the fluents will be the ſame as the co-efficient of the term e^{h-1} in the ſerieſes (P) and (Q) divided by h.

This ſeries, when the tangent of a given arc is known, finds the arc whoſe tangent differs very little from the given tangent.

11. Let t be increaſed or diminiſhed by a quantity e in (P), where e bears a very ſmall ratio to any root or value of t in

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the equations $P=0$ or $\frac{1}{P}=0$; and the reſulting quantity expanded into a ſeries $a+be+ce^2+de^3+$ &c. proceeding according to the dimenſions of e, and this ſeries be multiplied reſpectively into $\dot{t}$ and $\dot{e}$; and the fluents of the reſulting fluxions found; then will the former differ from the latter by $\int a\dot{t}$; for the former will be $\int a\dot{t}+ae+\frac{\dot{a}}{2\dot{t}}\times e^2+\frac{\ddot{a}}{2\cdot 3\dot{t}^2}\times e^3+\frac{\dddot{a}e^4}{2\cdot 3\cdot 4\dot{t}^3}+$ &c., and the latter $ae+\frac{\dot{a}}{2\dot{t}}e^2+\frac{\ddot{a}e^3}{2\cdot 3\dot{t}^2}+$ &c.

2. In the ſame manner, if more than one variable quantities x, y, z, &c. are contained in (P), which are increaſed by ſmall increments or decrements α, β, γ, &c., may the increments or decrements of the quantity (P) be deduced from the incremential theorem.

Ex. Let the quantity (P) be $(a+bx^n)^m$, and a, b, x, n, and m be increaſed by very ſmall increments α, β, γ, δ, and ε; then will $(a+\alpha+\overline{b+\beta}\times\overline{x+\gamma}^{\,n+\delta})^{m+\varepsilon}=(a+bx^n)^m+(a+bx^n)^m\times\log.\overline{a+bx^n}\times\varepsilon+m\times(a+bx^n)^{m-1}\times(\alpha+\beta x^n+b(x^n\delta\times\log.x+nx^{n-1}\gamma))+$ &c.

The terms are to be ſo placed, according to the dimenſions of the increments or decrements, that the greateſt may firſt occur.

12. Let ſome compound quantities be increaſed by any ſmall quantities variable or invariable, but not the variable quantities contained in them; then reduce the given quantity into a ſeries proceeding according to the dimenſions of the ſmall quantities, and find the fluent, integral, &c. as required.

Ex. Let the given quantity be $\frac{x^m\dot{x}}{(x^n+a^n+\beta)^r}=\frac{x^m\dot{x}}{(x^n+a^n)^r}-$

$\frac{r\beta\dot{x}x^{m}}{(x^{n}+a^{n})^{r+1}} + r \cdot \frac{r+1}{2} \times \frac{\beta^{2}\dot{x}x^{m}}{(x^{n}+a^{n})^{r+2}} - r \cdot \frac{r+1}{2} \cdot \frac{r+2}{3} \times \frac{\beta^{3}\dot{x}x^{m}}{(x^{n}+a^{n})^{r+3}} +$ &c., where the quantity $x^{n}+a^{n}$ is increaſed by a ſmall quantity β; then will $\int \frac{x^{m}\dot{x}}{(x^{n}+a^{n}+\beta)^{r}} = \int \frac{x^{m}\dot{x}}{(x^{n}+a^{n})^{r}} - \frac{\beta}{n \times a^{n}} \left(\frac{x^{m+1}}{(x^{n}+a^{n})^{r}} - \overline{m+1-rn} \times \mathrm{P} \right) + \frac{r\beta^{2}}{2 \,.\, n \times a^{n}} \left(\frac{x^{m+1}}{(x^{n}+a^{n})^{r+1}} - (m+1-\overline{r+1}n) \times \mathrm{Q} \right)$ $\frac{r \,.\, \overline{r+1}\beta^{3}}{2 \,.\, 3 \,.\, n \times a^{n}} \left(\frac{x^{m+1}}{(x^{n}+a^{n})^{r+2}} - (m+1-\overline{r+2}n) \times \mathrm{R} \right) + \frac{r \,.\, \overline{r+1} \,.\, \overline{r+2} \,.\, \beta^{4}}{2 \,.\, 3 \,.\, 4 \,.\, n \times a^{n}} \left(\frac{x^{m+1}}{(x^{n}+a^{n})^{r+3}} - (m+1-\overline{r+3}n) \times \mathrm{S} \right)$ &c. The continuation of the ſeries is evident; the letters P, Q, R, S, &c. denote the fluents $\int \frac{x^{m}\dot{x}}{(x^{n}+a^{n})^{r}}, \int \frac{x^{m}\dot{x}}{(x^{n}+a^{n})^{r+1}}, \int \frac{x^{m}\dot{x}}{(x^{n}+a^{n})^{r+2}}$, &c.

From the length of the arc of an hyperbola or ellipſe given may be deduced by this ſeries the length of a correſpondent arc of an hyperbola or ellipſe, of which the equations expreſſing the relation between its abſciſſæ and ordinates differ only by very ſmall quantities from the equations expreſſing the relation between the abſciſſæ and ordinates of the former.

13. The ſame principles may be applied to the reſolution of algebraical, fluxional, incremential, &c. equations.

Ex. Let L*m*M, &c. (Tab. IV. fig. 1.) be a given circle, whoſe center is C and radius (r), and it be required to find an arc LM, ſo that the area LSM deſcribed round a given point S contained between the lines LS, SM, and the arc of the circle LM be equal to a given area (α).

Find an arc Lm = A nearly equal to the arc required LM, of which to radius 1 ſubſtitute s for the ſine, and c for the coſine, and write e for mM = LM − Lm and b for SC; then will $\mathrm{LM} \times \frac{r}{2} (\overline{\mathrm{A}+e} \times \frac{r}{2}) \pm \mathrm{SC} \times \frac{r}{2} \times$ ſin. s arc : $\mathrm{LM}\left(\frac{b}{2} \times (\mathrm{A}+e - \right.$

A + e

$\frac{\overline{A+e}^3}{2\cdot 3r^2}+\frac{\overline{A+e}^5}{2\cdot 3\cdot 4\cdot 5r^4}$ &c.)) $=\frac{A}{2}\times r\pm\frac{b}{2}\times rs+\overline{\frac{r}{2}\pm\frac{b}{2}}\times c\times e\mp\frac{b}{2r}\times\left(\frac{1}{2}\times se^2+\frac{1}{2\cdot 3r}\times ce^3-\frac{1}{2\cdot 3\cdot 4r^2}\times se^4-\frac{1}{2\cdot 3\cdot 4\cdot 5r^3}\times ce^5+\right.$ &c.$) = \alpha$; in this equation for $\frac{2\alpha - Ar\mp b\times rs}{r\pm bc}$ ſubſtitute π, and for $r\pm bc$ write t, and the equation reſulting will be $e\mp\frac{b}{2tr}se^2\mp\frac{b}{1\cdot 2\cdot 3tr^2}\times ce^3\pm\frac{b}{1\cdot 2\cdot 3\cdot 4tr^3}\times se^4\pm\frac{b}{1\cdot 2\cdot 3\cdot 4\cdot 5tr^4}\times ce^5\mp$ &c. $=\pi$. From it find e in terms proceeding according to the dimenſions of π, and there reſults $e=\pi\pm\frac{bs}{1\cdot 2tr}\times\pi^2\pm\frac{3b^2s^2\pm bct}{2\cdot 3t^2r^2}\times\pi^3\pm$&c.

This method of reſolving KEPLER's and other problems of cutting a given area deſcribed round a point, whether focus or not, in a circle, when an approximate ſufficiently near to the area to be found is given, will converge as ſwift as any known method.

The reſolution of this problem may be deduced ſomewhat different by the following methods. Let the letters b, r, α, denote the ſame quantities as before, and s be the ſine of the arc Lm to radius 1, and $s+o$ the ſine of the arc LM nearly = Lm; then will $\frac{r^2}{2}\int\frac{\dot{s}}{\sqrt{(1-\overline{s+o}^2)}}\pm\frac{br}{2}\times\overline{s+o}=\alpha$; reduce the fluxion $\frac{\dot{s}}{\sqrt{(1-\overline{s+o}^2)}}$ into a ſeries $P\dot{s}+Qo\dot{s}+Ro^2\dot{s}+So^3\dot{s}+$ &c. and find the fluents of the fluxions $Q\dot{s}$, $R\dot{s}$, $S\dot{s}$, &c., which let be B, C, D, &c., there will reſult the equation $o+\frac{C}{t}o^2+\frac{D}{t}o^3+$&c. $=\frac{2\alpha\mp brs-r\times A}{t}=\pi$, where $t=Br^2\pm br$ and A = arc Lm; find o in terms of a ſeries proceeding to the dimenſions of π, and conſequently $s+o$ the ſine of the arc LM required.

From ſimilar principles may be found the tangent of the arc required from the equation $\frac{r^2}{2}\int\frac{i}{1+(t+i)^2} \pm \frac{\overline{t+i} \times r \times b}{2\sqrt{(1+\overline{t+i}^2)}} = \alpha$, where the tangent t to rad. 1 of the arc $Lm = A$ is given. Let the fluent of the fluxion $\left(\frac{i}{1+(t+i)^2}\right) = A + Bi + Ci^2 + Di^3 + \&c.$; and $\frac{\overline{t+i} \times b}{\sqrt{(1+\overline{t+i}^2)}} = A' + B'i + C'i^2 + D'i^3 + \&c.$, the equation will become $i + \frac{rC \pm C'}{rB \pm B'} i^2 + \frac{rD \pm D'}{rB \pm B'} \times i^3 + \&c. = \frac{2\alpha - r^2 A \mp rA'}{(r^2 B \pm rB')} = \pi$; from this equation inveſtigate $i = \pi + P\pi^2 + Q\pi^3 + \&c.$, and thence $t + i$ the tangent required.

In like manner may be found the ſecant, coſine, &c. of the arc required.

14. The ſame principles may be applied to cut an area deſcribed round any given point in a given curve equal to an area α.

Let x be the abſciſs and y the ordinate of the given curve, and b the diſtance of the beginning of the abſciſs from the given points, and let (A) be the area of the curve deſcribed round the point S, when the abſciſs is x, which differs very little from the given area (α); to find the value $x + e$ of the abſciſs, when the area $= \alpha$.

Let $y = X$ a function of x, and in X for x write $x + e$, and reduce the reſulting quantity (X') into a ſeries $X + Be + Ce^2 + De^3 + \&c.$ proceeding according to the dimenſions of e; then will the area $\int y\dot{x} = \int X\dot{x} + e\int B\dot{x} + e^2\int C\dot{x} + e^3\int D\dot{x} + \&c. = A + he + ke^2 + le^3 + \&c.$, and conſequently $A + he + ke^2 + le^3 + \&c. \pm (b \pm \overline{x+e}) \times \frac{y}{2} = A + he + ke^2 + le^3 + \&c. + \frac{1}{2}(b \pm \overline{x+e}) \times (X + Be + Ce^2 + De^3 + \&c.) = A + \frac{1}{2}(b \pm x) \times X + (b + \overline{\tfrac{1}{2}b \pm x} \times$

$B+\frac{1}{2}X)e+(k+\frac{1}{2}(b\pm x)C+\frac{1}{2}B)e^2+(l+\frac{1}{2}\overline{b\pm x}\times D+\frac{1}{2}C)e^3+$ &c. $=\alpha$; find the value of e in a ſeries proceeding according to the dimenſions of $\frac{\alpha-A-\frac{1}{2}(b\pm x)\times X}{b+\frac{1}{2}\overline{b\pm x}\times B+\frac{1}{2}X}$, and $x+e$ will be the abſciſs required; X denotes the value of y, when the abſciſs $=x$.

From the ſame principles may the ordinate y, &c. be found.

This problem may be reſolved in the ſame manner, when X denotes an infinite ſeries deduced from an equation expreſſing the relation between the abſciſs and ordinate of the given curve.

If the given area α be the difference between two areas SPM (α') and SPQ $=\beta$ (fig. 2.); for α ſubſtitute $\alpha'-\beta$, and the operation will be the ſame as the preceding.

15. 1. Given any equations, of which the increments of the quantities contained in them can be found from each other, and given approximate values of each of the unknown quantities, which nearly correſpond to each other; to find approximations, which differ leſs from the quantities themſelves than the given ones.

Suppoſe each of the given approximate values to be increaſed or diminiſhed by ſmall increments or decrements, as e, o, i, &c. which are the approximations to be found; and from the given find the equations reſulting from this hypotheſis; and from theſe may be deduced, by ſimple equations, the approximations ſought e, i, o, &c. by neglecting in them all the powers of e, o, i, &c. except the ſimple ones, and all the products of them multiplied into each other; and conſequently the equations deduced will contain only given quantities and ſimple powers of the unknown e, i, o, &c. to be found.

2. When two or more (n) values of one (x) of the unknown quantities are nearly equal to its given approximate; then the

equation which finds the approximate to x will be a quadratic or an equation of n dimenſions.

3. The approximations found by this method will converge more or leſs, according as the approximations given are more or leſs near to one value of each of the unknown quantities than to the remaining ones, &c.

Theſe principles were printed in the Medit. Algebr. in the years 1768 and 1769.

LEMMA.

15. Let kab (fig. 3.) be a circle, whoſe center is o; and Po perpendicular to the plane of the circle; and the force of any corpuſcle in the circle on the particle P vary as the corpuſcle divided by the (n) power of its diſtance from the particle; to find the attraction of the circle kab on the particle P.

From the ſuppoſita the force of any ring contained between the neareſt concentric circles lcd and hef of which the center is o, on the particle P, will be as the area of the ring divided by the nth power of the diſtance Ph; and conſequently if $\mathrm{P}o = \mathrm{A}$, $oh = v$, $hl = \dot{v}$, and $p =$ periphery of a circle of which the radius is 1; the attraction of the ring on P will be as $\dfrac{p \times v\dot{v}}{(\mathrm{A}^2+v^2)^{\overline{\frac{n}{2}}}}$, and the force of it in the direction Po as $\dfrac{\mathrm{P}o}{\mathrm{P}h} \times \dfrac{pv\dot{v}}{(\mathrm{A}^2+v^2)^{\overline{\frac{n}{2}}}} = \dfrac{p\mathrm{A}v\dot{v}}{(\mathrm{A}^2+v^2)^{\overline{\frac{n+1}{2}}}}$; of which fluxion the fluent is $\dfrac{p\mathrm{A}}{\overline{1-n} \times (\mathrm{A}^2+v^2)^{\overline{\frac{n-1}{2}}}}$; and thence the force of the circle kab on the particle P will be as $\dfrac{p \times \mathrm{A}}{\overline{1-n}\mathrm{P}k^{n-1}} - \dfrac{p \times \mathrm{A}}{\overline{1-n}\mathrm{PO}^{n-1}} \left(\dfrac{p}{\overline{1-n}\mathrm{A}^{n-2}}\right)$.

Cor. 1. The force of the area contained between the two circles *kab* and *lcd* on the particle P will be as $\frac{pA}{\overline{1-nPk}^{\,n-1}} - \frac{pA}{\overline{1-nPl}^{\,n-1}}$.

15. Let a ſolid be generated by the rotation of a given curve round a line in the axis P*o* or P*o* produced, and A the diſtance of P from *o*, and x = diſtance of any circle generated by the rotation of a point in the curve round the axis P*o* from *o*; and y the ordinate to the given curve, of which the abſciſs is x; and the ſame things be ſuppoſed as before; then will the attraction of the ſolid on the point P vary as

$$\int\frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}(A-x)^{n-1}}\left(\frac{p\dot{x}}{\overline{n-1}(A-x)^{n-2}}\right)-\int\frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}\times\left(\overline{A-x}^{\,2}+y^2\right)^{\frac{n-1}{2}}}.$$

Let the given curve be a circle, of which the center is *o*, and radius t, and conſequently $y^2 = t^2 - x^2$, and $\int\frac{p\dot{x}}{\overline{n-1}(A-x)^{n-2}}$

$$-\int\frac{p\times\overline{A-x}\times\dot{x}}{\overline{n-1}\,.\left(\overline{A-x}^{\,2}+y^2\right)^{\frac{n-1}{2}}}=\frac{p}{\overline{n-1}\,.\,\overline{n-3}\times(A-x)^{n-3}}-$$

$$\int\frac{p\dot{x}\times\overline{A-x}}{\overline{n-1}\times\left(A^2+t^2-2Ax\right)^{\frac{n-1}{2}}}=\frac{p}{\overline{n-1}\,.\,\overline{n-3}(A-x)^{n-3}}-\frac{p}{n-1}\left(\frac{1}{n-3}+\right.$$

$$\left.\frac{A^2+t^2}{\overline{n-5}\,.\,\overline{n-3}A^2}-\frac{x}{\overline{n-5}A}=\frac{\overline{n-4}A^2+t^2-\overline{n-3}Ax}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\right)\times\left(A^2+t^2-2Ax\right)^{\frac{3-n}{2}}.$$

By ſubſtituting $-t$ and t for x in the preceding expreſſion the attraction of the whole globe on the particle P will be found $\frac{p}{n-1}\left(\frac{A^2+\overline{n-3}At+t^2}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\times\overline{A+t}^{\,3-n}\ (L)\ -\ \frac{A^2-\overline{n-3}At+t^2}{\overline{n-5}\,.\,\overline{n-3}\times A^2}\times\overline{A-t}^{\,3-n}\ (H)\right)$.

16. If the particle be ſituated within the globe, and conſequently A leſs than t or $-t$, and n be an odd number negative

or affirmative; or fraction, of which the numerator is odd; then the preceding reſolution will be phyſically juſt; for in this caſe, if the attraction on one ſide of the particle be affirmative, the attraction on the other will mathematically be negative, that is, phyſically oppoſite: but if n be an even number, or fraction, of which the numerator is an even and denominator an odd number, the mathematical ſolution will not agree to the phyſical; for in the former the force on both ſides will be affirmative, in the latter the forces will be oppoſite, and therefore phyſically the force in this caſe will vary as $L+H$, and not as $L-H$, which is the force in both the caſes when A is greater than t and $-t$. The ſame may be applied to the more general reſolution.

17. Let ABCD (fig. 4.) be a globe, of which the diameters AB and CD are ſituated at right angles to each other, and AHBL be a ſpheroid generated by the revolution of an ellipſe on its axis AB, to which let HL nearly equal to $CD=AB$ be the conjugate, and P a point in the axis BA produced; to find the attraction of the ring contained between the globe and the ſpheroid on the point P, on the ſuppoſition that the force of any corpuſcle in the ring on the particle P varies as the magnitude of the corpuſcle directly, and the nth power of its diſtance from the corpuſcle inverſely.

Let $AB=CD=2t$, $CD=HL\ (2c)+2CH=2c+2e$, and conſequently $OH=c=t-e$, where e has a very ſmall ratio to t, $oP=A$, $po=x$, and pM parallel to $CD=y$; then, by the preceding lemma, the attraction of the circle whoſe radius is pM on the point P will be $\frac{p\times\overline{A-x}}{\overline{n-1}\,Pp^{n-1}}-\frac{p\times\overline{A-x}}{\overline{n-1}\,PM^{n-1}}$; and in like manner the attraction of the circle, whoſe radius is pm, will be $\frac{p\times\overline{A-x}}{\overline{n-1}\,Pp^{n-1}} - \frac{p\times\overline{A-x}}{\overline{n-1}\,.\,Pm^{n-1}}$, and conſequently the attraction of the ring contained between the two circles is

$$\frac{p\times\overline{A-x}}{\overline{n-1}\,.\,Pm^{n-1}} - \frac{p\times\overline{A-x}}{\overline{n-1}\,.\,PM^{n-1}} = \frac{p}{n-1}\times\left(\frac{A-x}{\left(\overline{A-x}^2+\frac{\overline{t-e}^2}{t^2}\times\overline{t^2-x^2}\right)^{\frac{n-1}{2}}} - \frac{A-x}{\left(\overline{A-x}^2+t^2-x^2\right)^{\frac{n-1}{2}}}\right) = \frac{p}{n-1}\times\frac{\overline{n-1}\times\overline{A-x}\times\left(t-\frac{x^2}{t}\right)}{(A^2+t^2-2Ax)^{\frac{n+1}{2}}}\times e + \&c.\times e^2 + \&c. = p\times\left(t-\frac{x^2}{t}\right)\times\frac{A-x}{(A^2+t^2-2Ax)^{\frac{n+1}{2}}}\times e$$

nearly, which multiplied into $\dot{x}$, and the fluent of the reſulting fluxion found, it will be

$$p\times(A^2+t^2-2Ax)^{\frac{1-n}{2}}\times\left(\left(\frac{1}{\overline{n-7}\times tA}=d\right)x^3-\left(\frac{1}{\overline{n-5}\,t}+\frac{3d\times\overline{A^2+t^2}}{\overline{n-5}\,A}=c\right)x^2-\left(\frac{t}{\overline{n-3}\,A}+2c\times\frac{A^2+t^2}{\overline{n-3}\times A}=b\right)x+\frac{t}{n-1}-\frac{b\times\overline{A^2+t^2}}{\overline{n-1}\,A}\right)e\;(M).$$

If the attraction of the whole ring contained between the ſphere and ſpheroid be required, ſubſtitute in the fluent (M) for x, t and $-t$, and proceed as in the preceding caſes; in like manner may be found the attraction of the ring contained between any two values of x.

18. The ſame principles may be applied to find the attraction of the above-mentioned ring, when the line Pp is not perpendicular to the circles pM, pm, CD, &c., and does not cut the diameters CD, M′M, &c. into two equal parts. They may be further applied for finding the attraction of rings contained between any other given ſolids, of which the equations differ by very ſmall quantities from each other; for example,

between two ſpheroids, of which the axes in the one do not differ much from correſpondent axes in the other, in which caſe the fluents found of the following fluxions may be uſeful.

$$1.\ \int\frac{\dot{x}}{(t^{2}-x^{2})^{\frac{\overline{n-1}}{2}}}=\frac{1}{\overline{n-3}\,t^{2}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-3}}{2}}}+\frac{n-4}{\overline{n-3}\,.\,\overline{n-5}\,t^{4}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-5}}{2}}}$$

$$+\frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,t^{6}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-7}}{2}}}+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,t^{8}}\times$$

$$\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-9}}{2}}}\ .\,.\,.\,.\,.\ \frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,.\,.\,3}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,.\,.\,2t^{n-3}}\times\frac{x}{t^{2}-x^{2}}+$$

$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,.\,.\,3\,.\,1}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,.\,2t^{n-3}}\times\int\frac{\dot{x}}{t^{2}-x^{2}}$ if n be an odd number; but

$$2.\ \int\frac{\dot{x}}{(t^{2}-x^{2})^{\frac{\overline{n-1}}{2}}}=\frac{1}{\overline{n-3}\,t^{2}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-3}}{2}}}+\frac{n-4}{\overline{n-3}\,.\,\overline{n-5}\,t^{4}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-5}}{2}}}$$

$$+\frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,t^{6}}\times\frac{x}{(t^{2}-x^{2})^{\frac{\overline{n-7}}{2}}}\ .\,.\,.\,.\,.\ +$$

$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\,.\,.\,.\,.\,4\,.\,2}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\,.\,.\,.\,.\,3\,.\,1t^{n-2}}\times\frac{x}{(t^{2}-x^{2})^{\frac{1}{2}}}$, if n be even.

Theſe principles may be applied to the finding approximations in very many philoſophical problems.

Cor. 1. From hence may be deduced the ſubſequent arithmetical theorems.

$$1.\ \frac{1}{2m-1}+\frac{2m-2}{\overline{2m-1}\,.\,\overline{2m-3}}+\frac{\overline{2m-2}\,.\,\overline{2m-4}}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}}+$$

$$\frac{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}}\ .\,.\,.\,.\,.\ \frac{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}\,.\,.\,.\,.\,2}{\overline{2m-1}\,.\,\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,.\,.\,3\,.\,1}$$

$= 1$; or, which is the ſame, $\frac{1}{2m-3}+\frac{2m-4}{\overline{2m-3}\,.\,\overline{2m-5}}+$

2m − 4

$$\frac{\overline{2m-4}\times\overline{2m-6}}{\overline{2m-3}\times\overline{2m-5}\times\overline{2m-7}}\ldots\ldots+\frac{\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\ldots4\times2}{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots3\times1}$$

$$=1,\text{ or }\frac{1}{2m-5}+\frac{2m-6}{\overline{2m-5}\,.\,\overline{2m-7}}\ldots+\frac{\overline{2m-6}\,.\,\overline{2m-8}\ldots4\times2}{\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots3\times1}$$

$=1$, &c.

2. $$\frac{1}{2m-2}+\frac{2m-3}{\overline{2m-2}\,.\,\overline{2m-4}}+\frac{\overline{2m-3}\,.\,\overline{2m-5}}{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}}+$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}}{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}}\ldots\ldots+$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots7\,.\,5}{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\,.\,\overline{2m-10}\ldots6\,.\,4}+2\times$$

$$\frac{\overline{2m-3}\,.\,\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots3}{\overline{2m-2}\,.\,\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\,.\,\overline{2m-10}\ldots2}=1,\text{ or}$$

$$\frac{1}{2m-4}+\frac{2m-5}{\overline{2m-4}\,.\,\overline{2m-6}}+\frac{\overline{2m-5}\,.\,\overline{2m-7}}{\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}}\ldots\ldots+$$

$$\frac{\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\ldots5}{\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\ldots4}+2\times$$

$$\frac{\overline{2m-5}\,.\,\overline{2m-7}\,.\,\overline{2m-9}\,.\,\overline{2m-11}\ldots5\,.\,3}{\overline{2m-4}\,.\,\overline{2m-6}\,.\,\overline{2m-8}\,.\,\overline{2m-10}\,.\,\overline{2m-12}\ldots4\,.\,2}=1,\text{ &c.}$$

3. By expanding the terms of the preceding fluents may be deduced the arithmetical theorems, *viz.*

1. $$\frac{1}{n-3}\times\overline{n-3}\times\overline{n-1}\,.\,\overline{n+1}\,.\,\overline{n+3}\ldots\ldots\overline{n+m}+\frac{n-4}{\overline{n-3}\,.\,\overline{n-5}}$$

$$\times\,\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\,.\,\overline{n+1}\ldots\overline{n+m-2}+\frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}}$$

$$\times\,\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\ldots\overline{n+m-4}+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}}\times\overline{n-9}\,.\,\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\ldots\overline{n+m-6}$$

$$+\ldots\ldots+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots\ldots3}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\ldots2}\times2\,.\,4\,.\,6\,.\,8\ldots$$

$$\overline{m+5}+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\,.\,\overline{n-10}\ldots3}{\overline{n-3}\,.\,\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\,.\,\overline{n-11}\ldots2}\times2\times4\times6\times8\ldots$$

$m+5$

$$\overline{m+5}\times\frac{1}{m+6}=\overline{n-1}\,.\,\overline{n+1}\,.\,\overline{n+3}\,.\,\overline{n+5}\ldots\overline{n+m+2}\times\frac{1}{m+6}$$

if n be an odd number.

$$2.\ \frac{1}{n-3}\times\overline{n-3}\,.\,\overline{n-1}\,.\,\overline{n+1}\,.\,\overline{n+3}\ldots\overline{n+2m+1}+$$

$$\frac{n-4}{\overline{n-3}\cdot\overline{n-5}}\times\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\,.\,\overline{n+1}\ldots\overline{n+2m-1}+$$

$$\frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\,.\,\overline{n-1}\ldots\overline{n+2m-3}+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\,.\,\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-9}\,.\,\overline{n-7}\,.\,\overline{n-5}\,.\,\overline{n-3}\ldots\overline{n+2m-5}$$

$$\ldots+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}\ldots2}{\overline{n-3}\cdot\overline{n-5}\,.\,\overline{n-7}\,.\,\overline{n-9}\ldots3\times1}\times1\cdot3\cdot5\cdot7\ldots\overline{2m+5}=$$

$$\overline{n-1}\,.\,\overline{n+1}\,.\,\overline{n+3}\,.\,\overline{n+5}\ldots\overline{n+2m+3}\times\frac{1}{2m+7},$$ if n be an even number.

4. Arithmetical theorems, ſomewhat different, may be deduced from taking the fluxions of the preceding fluents, and reducing the fractions reſulting to a common denominator by multiplying them into t^2-x^2, $(t^2-x^2)^2$, $(t^2-x^2)^3$, &c.

$$1.\ \frac{1}{n-3}\times\overline{n-4}+\frac{n-4}{\overline{n-3}\cdot\overline{n-5}}\times\overline{n-7}+\frac{\overline{n-4}\,.\,\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-10}$$

$$+\frac{\overline{n-4}\,.\,\overline{n-6}\,.\,\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-13}\ldots+\frac{\overline{n-4}\cdot\overline{n-6}\,.\,\overline{n-8}\ldots3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots2}$$

$$\times\frac{7-n}{2}-\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\ldots3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\,.\,\overline{n-11}\ldots2}\times\frac{n-3}{2}=0,$$ if n be odd.

$$2.\ \frac{1}{n-3}\times\overline{n-4}+\frac{n-4}{\overline{n-3}\,.\,\overline{n-5}}\times\overline{n-7}+\frac{\overline{n-4}\cdot\overline{n-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}}\times\overline{n-10}$$

$$+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}}\times\overline{n-13}\ldots+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots2}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots3\times1}$$

$\times\frac{4-n}{2}=0$, if n be even; or more general, if n be even,

$$3.\quad \frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots\overline{n-2m-2}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots\overline{n-2m-3}}\times\overline{n-2m-4}+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\ldots\overline{n-2m-4}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\cdot\overline{n-11}\ldots\overline{n-2m-5}}\times(\overline{m+1}\times\overline{n-2m-6}-1)$$

$$+\ \frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\cdot\overline{n-12}\ldots\overline{n-2m-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\cdot\overline{n-11}\cdot\overline{n-13}\ldots\overline{n-2m-7}}\times$$

$$(\overline{m+1}\cdot\frac{m+2}{2}\times\overline{n-2m-8}-\overline{m+2})+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\cdot\overline{n-12}\cdot\overline{n-14}\ldots\overline{n-2m-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\cdot\overline{n-11}\cdot\overline{n-13}\cdot\overline{n-15}\ldots\overline{n-2m-9}}$$

$$\times\ (\overline{m+1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\times\overline{n-2m-10}-\overline{m+2}\cdot\frac{m+3}{2})\ +$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\cdot\overline{n-10}\ldots\overline{n-2m-10}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\cdot\overline{n-11}\ldots\overline{n-2m-11}}\times(\overline{m+1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\cdot$$

$$\frac{m+4}{4}\times\overline{n-2m-12}-\overline{m+2}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3})\ \ldots\ldots+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots 2}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\ldots 3\times 1}\times-\overline{m+2}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\ldots\ldots\frac{n-4}{n-2m-6}$$

$$=0.$$

$$4.\quad \frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots\overline{n-2m-2}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots\overline{n-2m-3}}\times\overline{n-2m-4}+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots\overline{n-2m-4}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots\overline{n-2m-5}}\times(\overline{m+1}\times\overline{n-2m-6}-1)+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots\overline{n-2m-6}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots\overline{n-2m-7}}\times(\overline{m+1}\cdot\frac{m+2}{2}\times\overline{n-2m-8}-$$

$$\overline{m+2})+\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots\overline{n-2m-8}}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots\overline{n-2m-9}}\times(\overline{m+1}\cdot\frac{m+2}{2}\times$$

$$\frac{m+3}{3}\times\overline{n-2m-10}-\overline{m+2}\cdot\frac{m+3}{2})+\ldots\ldots\ldots+$$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots 5\times 3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots 4\times 2}\times(\overline{m+1}\cdot\frac{m+2}{2}\cdot\frac{m+3}{3}\ldots\frac{n-5}{n-2m-5}-$$

$$\overline{m+2}\cdot\frac{m+3}{2}\cdot\frac{m+4}{3}\ldots\frac{n-5}{n-2m-7})-$$

$n-4$

$$\frac{\overline{n-4}\cdot\overline{n-6}\cdot\overline{n-8}\ldots5\times3}{\overline{n-3}\cdot\overline{n-5}\cdot\overline{n-7}\cdot\overline{n-9}\ldots4\times2}\times\overline{m+2}\cdot\frac{m+3}{2}\times\frac{m+4}{3}\ldots$$

$\frac{n-3}{n-2m-5}=0$, if n be an odd number.

Many more arithmetical theorems may be deduced from the fluents of theſe and other fluxions by ſimilar methods, which cannot, without ſome difficulty, be found from the common methods of finding the ſums of ſeries.

This method of finding approximations to the areas and lengths of curves, fluents of fluxions, and ſums of ſeries, &c. of which the equations, to their increments, fluxions, &c. are given from the areas of curves, fluents, &c. of which the equations to their increments, &c. differ by very ſmall quantities from the given equations was publiſhed in the Meditat. Analyt. near twenty years ago.

I ſhall conclude this Paper with two theorems of ſome little uſe in the doctrine of chances.

THEOREM I.

$$H=\overline{a+b}\times\overline{a+b-1}\cdot\overline{a+b-2}\cdot\overline{a+b-3}\ldots\overline{a+b-n+1}$$
$$=a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+1}+n\cdot a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a\quad n+2}$$
$$\times b+n\cdot\frac{n-1}{2}\times a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+3}\times b\cdot\overline{b-1}+n\cdot$$
$$\frac{n-1}{2}\cdot\frac{n-2}{3}a\cdot\overline{a-1}\cdot\overline{a-2}\ldots\overline{a-n+4}\times b\times\overline{b-1}\cdot\overline{b-2}+$$
$$\ldots\ldots+n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\cdot\frac{n-3}{4}\ldots\ldots\frac{n-l+1}{l}\,(\mathrm{L})\;a\cdot\overline{a-1}\cdot\overline{a-2}$$
$$\cdot\overline{a-3}\ldots\overline{a-n+l+1}\times b\cdot\overline{b-1}\cdot\overline{b-2}\ldots\overline{b-l+1}+\ldots+$$

$n \,.\, \frac{n-1}{2}\, a \,.\, \overline{a-1} \,.\, b \,.\, \overline{b-1} \,.\, \overline{b-2} \,.\,.\, \overline{b-n+3} + na \,.\, b \,.\, \overline{b-1} \,.\, \overline{b-2} \,.\,.\,.\, \overline{b-n+2} + b \,.\, \overline{b-1} \,.\, \overline{b-2} \,.\,.\,.\, \overline{b-n+1}$.

If for $a+b-1$, $a+b-2$, $a+b-3$, &c., $a-1$, $a-2$, &c., $b-1$, $b-2$, &c. be ſubſtituted reſpectively $a+b-x$, $a+b-2x$, $a+b-3x$, &c., $a-x$, $a-2x$, $a-3x$, &c., $b-x$, $b-2x$, $b-3x$, &c., the reſulting equation will equally be juſt; and, laſtly, if for x be ſubſtituted 0, it will become the binomial theorem.

Cor. If there are two different events A and B, of which the numbers are reſpectively a and b, and their chances of happening alſo as a and b; and if A's happen, let the whole number $(a+b)$ and alſo the number of A's be diminiſhed by x, and in the ſame manner of B's happening, and ſo on; then will the chance of A's happening $n-l$ times, and B's happening l times in n trials be $\mathrm{L} \times a \,.\, \overline{a-x} \,.\, \overline{a-2x} \,.\,.\, \overline{a-(n-l-1)x} \times b \,.\, \overline{b-x} \,.\, \overline{b-2x} \,.\,.\, \overline{b-(l-1)x}$ divided by H.

In a ſimilar manner may be found, 1. the chance of A's happening between h and k times; and, 2. the chance of A's happening (h) to B's happening (k) times; 3. of A's and B's happening reſpectively h and k times more than the other; 4. the chance of A's happening an even to its happening an odd number of times, &c. in (n) trials, &c. &c. &c.

THEOREM II.

$$\mathrm{H} = \overline{a+b+c+d+\&c.} \times \overline{a+b+c+d+\&c.-x} \times \overline{a+b+c+d+\&c.-2x} \,.\,.\,.\,.\, \overline{a+b+c+d+\&c.-(n-1)x} = a \,.\, \overline{a-x} \,.\, \overline{a-2x} \,.\,.\,.\, \overline{a-\overline{n-1}x} + n \,.\, a \,.\, \overline{a-x} \,.\, \overline{a-2x} \,.\,.\, \overline{a-\overline{n-2}x} \times \overline{b+c+d+\&c.}$$

$$+n\cdot\frac{n-1}{2}\cdot a\cdot\overline{a-x}\cdot\overline{a-2x}\,..\,\overline{a-\overline{n-3}x}\times(b\cdot\overline{b-x}+c\times\overline{c-x}+d\cdot\overline{d-x}+\&c.+2bc+2bd+2cd+\&c.)+\,\ldots\,+\mathrm{L}\times(a\cdot\overline{a-x}\cdot\overline{a-2x}\,\ldots\,\overline{a-\overline{l-1}x}\times b\cdot\overline{b-x}\cdot\overline{b-2x}\,..\,\overline{b-\overline{m-1}x}\times c\cdot\overline{c-x}\cdot\overline{c-2x}\,..\,\overline{c-\overline{p-1}x}\times d\cdot\overline{d-x}\cdot\overline{d-2x}\,\ldots\,\overline{d-\overline{q-1}x}\times\&c.=\mathrm{K})+\&c.,$$

where $\mathrm{L}=n\cdot\frac{n-1}{2}\cdot\frac{n-2}{3}\,..\,\frac{n-l+1}{l}\times\overline{n-l}\cdot\frac{n-l-1}{2}\cdot\frac{n-l-2}{3}\,..\,\frac{n-l-m+1}{m}\times\overline{n-l-m}\cdot\frac{n-l-m-1}{2}\cdot\frac{n-l-m-2}{3}\,\ldots\,\frac{n-l-m-p+1}{p}\times\overline{n-l-m-p}\cdot\frac{n-l-m-p-1}{2}\cdot\frac{n-l-m-p-2}{3}\,..\,\frac{n-l-m-p-q+1}{q}\times\&c.$ which is the ſame as the co-efficient of the term $a^l\times b^m\times c^p\times d^q\times\&c.$ in the multinomial $a+b+c+d+\&c.$ raiſed to the power n.

The chance of any number of events A, B, C, D, &c. of which the numbers are a, b, c, d, &c. happening l, m, p, q, &c. times reſpectively in a ſimilar manner to A's and B's happening in the preceding caſe will be $\frac{\mathrm{L}\times\mathrm{K}}{\mathrm{H}}$.

All the propoſitions mentioned as immediately deducible from the preceding theorem may, *mutatis mutandis*, with the ſame eaſe be applied to more events A, B, C, D, &c.

If for a, b, c, d, &c. be ſubſtituted the ſame letters, increaſed or diminiſhed by any given quantities, the reſulting equation will be equally true.

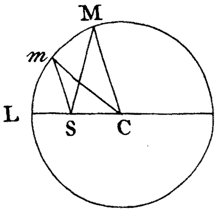

Fig 1.

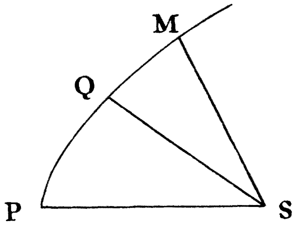

Fig 2.

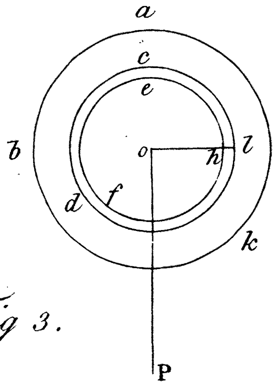

Fig 3.

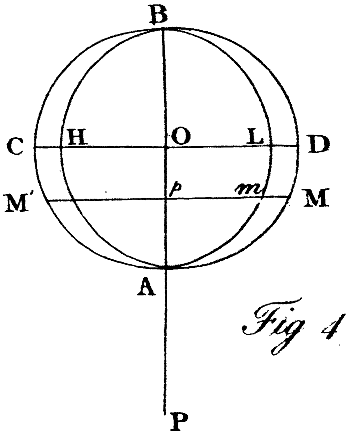

Fig 4.